A VISCOPLASTIC BOUNDARY LAYER WITH A POWER-LAW VELOCITY DISTRIBUTION ALONG THE OUTER BOUNDARY

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We consider the self-similar solution of the problem of the stationary viscoplastic boundary layer which arises during the flow of a plastic material past a rough solid wall in the case of a power-law velocity distribution along the outer boundary.

1. The defining equation of a viscoplastic medium is [1]

$$P_{ij} = -p\delta_{ij} + \left(2\mu + \frac{k}{\sqrt{V_{(2)}}}\right)V_{ij} \tag{1.1}$$

where P_{ij} is the stress tensor, p is the pressure, δ_{ij} is a unit tensor, μ is the coefficient of viscosity, k is the yield strength, and V_{ij}, $V_{(2)}$ the straining rate tensor and its second basic invariant, respectively. In the case of plane flow of an incompressible material $V_{(2)} = V_{11}^2 + V_{12}^2$.

We assume that the yield strength is sufficiently large, so that

$$B \gg R^{3/2} \quad (B = kL / \mu U) \tag{1.2}$$

Here B is the Bingham number, R is the Reynolds number, and L and U are the characteristic linear dimension and velocity, respectively.

Condition (1.2) enables us to neglect the inertial terms and to write the boundary-layer equations in the form [1, 2]

$$\frac{\partial p}{\partial x_1} = \mu\frac{\partial^2 v}{\partial x_2^2} + 2\omega k\left(\frac{\partial \eta}{\partial x_1} - 2\eta\frac{\partial \eta}{\partial x_2}\right), \qquad \frac{\partial p}{\partial x_2} = -2\omega k\frac{\partial \eta}{\partial x_2} \tag{1.3}$$

$$\eta = \frac{\partial v/\partial x_1}{\partial v/\partial x_2}, \qquad \omega = \operatorname{sign}\frac{\partial v}{\partial x_2} \tag{1.4}$$

The axis x_1 is directed along the solid wall; the axis x_2 is perpendicular to the wall. We assume that the medium is ideally plastic outside the boundary layer.

Another type of viscoplastic boundary layer is considered in [3].

Let the velocity distribution in the exterior flow near the boundary layer be described by a power law, and let the pressure over some interval of x_1 values be approximately constant.

We assume that the shearing stress at the upper boundary of the layer is equal to the yield strength of the material. Neglecting slippage at the lower boundary of the layer, we have the following boundary conditions:

$$\begin{gathered} v(x_1, 0) = 0, \qquad v(x_1, \delta) = \omega U (x_1 / L)^n \\ p(x_1, \delta) = p_0 = \text{const}, \qquad P_{21}(x_1, \delta) = \omega k \end{gathered} \tag{1.5}$$

where $\delta = \delta(x_1)$ is the thickness of the boundary layer.

Let us introduce the dimensionless variables

$$x_1^* = \frac{x_1}{L}, \quad x_2^* = \frac{x_2}{\varepsilon L}, \quad v^* = \frac{v}{U}, \quad p^* = \frac{p - \varepsilon k}{\varepsilon k}, \quad \varepsilon = B^{-1/3} \ll 1 \tag{1.6}$$

Equations (1.3) now become

$$\frac{\partial f^*}{\partial x_1^*} = \frac{\partial^2 v^*}{\partial x_2^{*2}} + 4\omega\left(\frac{\partial \eta^*}{\partial x_1^*} - \eta^* \frac{\partial \eta^*}{\partial x_2^*}\right), \qquad \eta^* = \frac{\partial v^*/\partial x_1^*}{\partial v^*/\partial x_2^*}$$

$$p^* = -2\omega\eta^* + f^* + p_0^* \tag{1.7}$$

where f^* is an arbitrary function of x_1^*. By virtue of (1.1) and (1.6), to within terms of the order of $k\varepsilon^4$ we have

$$P_{21} = k\left[\omega + \varepsilon^2\left(\frac{\partial v^*}{\partial x_2^*} - 2\omega\eta^{*2}\right)\right] \tag{1.8}$$

Conditions (1.5) with applowance for (1.8) become

$$v^*(x_1^*, 0) = 0, \ v^*(x_1^*, \delta^*) = \omega x_1^{*n}, \ p^*(x_1^*, \delta^*) = p_0^* \tag{1.9}$$

$$\partial v^*/\partial x_2^* - 2\eta^{2*} = 0 \qquad \text{for} \quad x_2^* = \delta^*$$

2. We shall obtain the expression for the velocity in the layer in the form

$$v = \omega x_1^n \varphi(\xi), \qquad \xi = x_2/\delta, \qquad \delta = D x_1^m \tag{2.1}$$

(The asterisks used above to denote dimensionless quantities will be omitted from now on.)

From (1.7) and (1.9) we obtain (2.2)

$$\varphi''(\varphi'^3 + 4n^2D^3\varphi^2) - {}^4/_9\,(n-1)\,D^3\varphi'^2\,[(2+n)\,\xi\varphi' + 3n\varphi] = {}^1/_3\,(n-1)\,\omega\, CD^2\varphi'^3$$

$$p = p_0 + \Phi(\xi)\, x_1^{(n-1)/3}, \qquad \Phi(\xi) = C - 2\omega D\,(n\,\varphi\,/\,\varphi' - m\xi) \tag{2.3}$$

$$m = {}^1/_3\,(2+n), \ n \neq 1 \tag{2.4}$$

$$\varphi(0) = 0, \quad \varphi(1) = 1, \quad \Phi(1) = 0 \quad \beta^3 - 2D^3(n - m\beta)^2 = 0, \quad \beta = \varphi'(1) \tag{2.5}$$

By virtue of (2.3) and (2.5), the constant C is given by

$$C = 2\omega\,(n - m\beta)\,D\,/\,\beta \tag{2.6}$$

Thus, in satisfying the boundary condition for the pressure, we must renounce the condition $\beta = 0$, i.e. the smooth matching of the solutions inside and outside the boundary layer. The case $n = 0$ considered in [2] is an exception. Here (2.2), (2.4) and (2.6) yield

$$\varphi'' = {}^4/_9 D^3(1 - 2\xi), \qquad m = {}^2/_3, \qquad C = -{}^4/_3\omega D \tag{2.7}$$

which coincides with the results of the latter paper.

Integrating (2.7) under conditions (2.5) and $\beta = 0$, we obtain

$$\varphi' = 6\xi(1-\xi), \quad \varphi = \xi^2(3 - 2\xi), \quad D = 2{,}381 \tag{2.8}$$

This solution is characterized by the fact that

$$\varphi'(0) = 0 \tag{2.9}$$

and by the presence of an inflection point of the function $\varphi(\xi)$ in the middle of the layer.

From (1.1) and (2.9) we infer that the shearing stress at the wall is exactly equal to the yield strength of the material. As is noted in [2], this result means that the theory under consideration is valid only for very small velocities of the viscoplastic medium.

Now let us consider the case of an arbitrary n (not equal to unity). We can solve the problem by the method proposed by Shvets [4] for an ordinary viscous boundary layer.

Taking $\varphi = \xi$ as our zeroth approximation, we integrate twice to obtain our approximate solution of Eq. (2.2),

$$\varphi'' = \frac{n-1}{9} D^2 \frac{3\omega C + 8D(1+2n)\xi}{1 + 4a^2\xi^2} \qquad (a^2 = n^2D^3) \tag{2.10}$$

$$\varphi' = \frac{n-1}{9n^2}\left[\frac{3\omega Ca}{2D}\operatorname{arc\,tg} 2a\xi + (1+2n)\ln(1 + 4a^2\xi^2)\right] + \alpha \qquad (\alpha = \varphi'(0)) \tag{2.11}$$

$$\varphi=\frac{n-1}{9n^2}\left\{\left[\frac{3\omega Ca}{2D}\xi+\frac{1}{a}(1+2n)\right]\operatorname{arc\,tg}2a\xi+\right.\tag{2.12}$$
$$\left.+\left[(1+2n)\xi-\frac{3\omega C}{8D}\right]\ln(1+4a^2\xi^2)\right\}+\left[\alpha-\frac{2}{9n^2}(n-1)(1+2n)\right]\xi$$

From the second condition of (2.5) with allowance for (2.6) we obtain

$$\frac{9n^2}{n-1}\beta^2-\left[\frac{13n^2-2n-2}{n-1}-\frac{1}{a}(1+2n)\operatorname{arc\,tg}2a-\right.\tag{2.13}$$
$$\left.-{}^1/_4(2+n)\ln(1+4a^2)\right]\beta-{}^3/_4 n\ln(1+4a^2)=0$$

and from Eq. (2.11) we have

$$\alpha=\beta-\frac{n-1}{9n^2}\left[3a\left(\frac{n}{\beta}-m\right)\operatorname{arc\,tg}2a+(1+2n)\ln(1+4a^2)\right]\tag{2.14}$$

Adding the fourth condition of (2.5) to (2.13), (2.14), we obtain the complete system of equations for determining α, β and D.

Numerical solution of this system for several values of the exponent n yields the following results:

$n=-3$	-2	-1	2	3
$\alpha=0.415$	0.449	0.548	0.352	0.332
$\beta=1.426$	1.423	1.379	1.273	1.332
$D=0.611$	0.712	0.851	2.238	1.247

The values of the function $\varphi(\xi)$ are given in Table 1; Fig. 1 shows the curves of this function.

Table 1

ξ	$n=-3$	$n=-2$	$n=-1$	$n=2$	$n=3$
0.0	0.000	0.000	0.000	0.000	0.000
0.1	0.047	0.050	0.059	0.048	0.043
0.2	0.108	0.113	0.128	0.120	0.109
0.3	0.183	0.188	0.209	0.207	0.191
0.4	0.271	0.276	0.295	0.304	0.286
0.5	0.371	0.375	0.393	0.408	0.390
0.6	0.481	0.484	0.499	0.518	0.501
0.7	0.600	0.602	0.614	0.633	0.619
0.8	0.727	0.728	0.736	0.752	0.741
0.9	0.860	0.861	0.865	0.874	0.869
1.0	1.000	1.000	1.000	1.000	1.000

Numerical integration of Eq. (2.2) under conditions (2.5) for given values of n indicates that the error of the results obtained by means of approximate formula (2.12) does not exceed 4%.

The transverse pressure distribution is characterized by the function $\Phi(\xi)$. The values of this function are given in Table 2; it is plotted in Fig. 2.

The calculations were carried out for convergent flow ($\omega=-1$) in the case of negative values of n.

The case $n=-2$ corresponds to the motion of an ideally plastic material in a conical nozzle (see [5], p. 91). Motion in a flat convergent channel can be approximated by setting $n = -1$.

If the velocity distribution in the exterior stream is not given by a power law, then we can use the method of Smith [6, 7] which consists in choosing a power-law velocity distribution which approximates the given distribution over some interval.

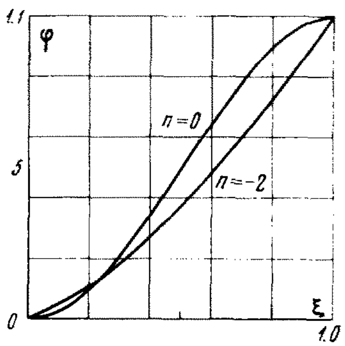

Fig. 1

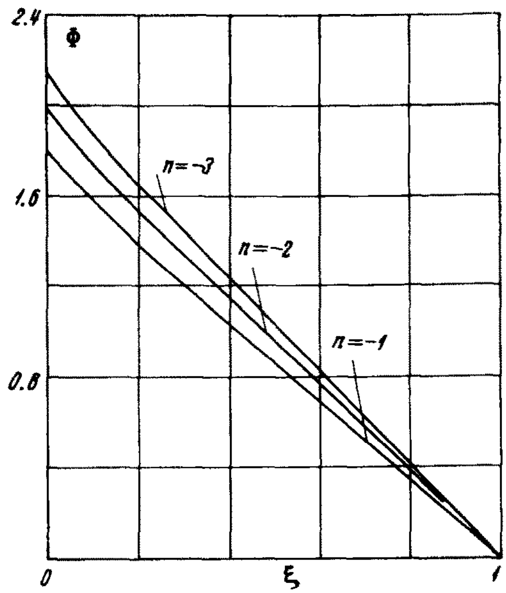

Fig. 2

Table 2

ξ	$n = -3$	$n = -2$	$n = -1$	$n = 2$	$n = 3$
0.0	2.162	2.000	1.801	1.066	1.462
0.1	1.880	1.745	1.586	0.965	1.286
0.2	1.659	1.534	1.391	0.923	1.205
0.3	1.462	1.344	1.210	0.849	1.102
0.4	1.269	1.160	1.035	0.754	0.976
0.5	1.071	0.976	0.863	0.646	0.834
0.6	0.867	0.788	0.693	0.528	0.681
0.7	0.658	0.597	0.521	0.403	0.520
0.8	0.443	0.401	0.349	0.273	0.351
0.9	0.223	0.202	0.175	0.138	0.178
1.0	0.000	0.000	0.000	0.000	0.000

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BOUNDARY VALUE PROBLEMS FOR TURBULENCE MODEL EQUATIONS

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Boundary value problems for a system of equations serving as a mathematical model of the turbulent motion of a liquid or gas are investigated. The model in question was introduced by Burgers in [2]. Section 1 contains a proof of the existence of at least one smooth time-periodic solution of the first boundary value problem for this system. This is accomplished with the aid of the Leray-Schauder topological principle [2] concerning the existence of fixed points of completely continuous operators. The existence theorem is prefaced by a derivation of the prior estimates of the solution of the problem which are necessary for the realization of the topological principle. Section 2 deals with the first boundary value problem with initial conditions and with the Cauchy problems for the turbulence model equations.

Let us begin by introducing some symbols. We denote the interval $(0, 1)$ by Ω. Let $t_1, t_2 \in (-\infty, \infty)$ and let $t_2 > t_1$. The symbol $Q_{t_1, t_2} = \Omega \times (t_1, t_2]$ denotes the rectangle. If $t_1 = -\infty$ and $t_2 = +\infty$, then the rectangle Q_{t_1, t_2} becomes a strip which we denote by Q. Every rectangle for which $t_2 - t_1 = \tau_0$, where τ_0 is a fixed number, will be denoted by Q_{τ_0}. From now on we shall assume that $t_1 = 0$ and $t_2 = T$. The closures of Q_{t_1, t_2}, Q and Q_{τ_0} will be denoted by $\overline{Q}_{t_1, t_2}$, $\overline{Q}$ and $\overline{Q}_{\tau_0}$.

The scalar product in the space L_2 of functions in Q_{τ_0} and the norm are given by the expressions

$$(\Phi_1, \Phi_2)_{Q_{\tau_0}} = \int_0^{\tau_0}\int_0^1 \Phi_1\Phi_2 \, dx\, dt, \qquad \|\Phi\|_{Q_{\tau_0}}{}^2 = \int_0^{\tau_0}\int_0^1 \Phi^2 \, dx\, dt$$

The scalar product and the norm in L_2 for every $t \in [0, \tau_0]$ will be denoted in similar fashion,

$$(\Phi_1, \Phi_2)_{\Omega} = \int_0^1 \Phi_1\Phi_2 \, dx, \qquad \|\Phi\|_{\Omega}{}^2 = \int_0^1 \Phi^2 \, dx$$

The Hölder norms for the function $\Phi(x, t)$ defined in Q_{t_1, t_2} are defined as follows:

$$|\Phi|_0 = \sup|\Phi|, \quad |\Phi|_\alpha = |\Phi|_0 + \sup\frac{|\Phi(P_2) - \Phi(P_1)|}{[d(P_1, P_2)]^\alpha}$$

$$|\Phi|_{1+\alpha} = |\Phi|_\alpha + |\Phi_x|_\alpha, \qquad |\Phi|_{2+\alpha} = |\Phi|_{1+\alpha} + |\Phi_t|_\alpha + |\Phi_{xx}|_\alpha$$